SOLUTION TO THE PROBLEM OF TRANSIENT ASYMMETRIC HEAT CONDUCTION IN A TWO-LAYER HOLLOW CYLINDER OF FINITE LENGTH

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The problem is solved by the method of finite integral transformations with asymmetric initial and boundary conditions, when heat is generated.

We consider the problem of determining the temperature distribution in a two-layer hollow cylinder of finite length with an arbitrary mode of heat generation both inside the volume of the cylinders and at their interface, with power densities

$$Q = \begin{cases} Q_i & \text{at } r \in (r_{i-1}, r_i), \ 0 \leqslant \varphi \leqslant 2\pi, \ 0 < z < l, \ t > 0, \\ S & \text{at } r = r_1, \ 0 \leqslant \varphi \leqslant 2\pi, \ 0 < z < l, \ t > 0 \end{cases}$$

$$(i = 1;\ 2).$$

The solution to this problem must satisfy the equation

$$a_i^2 \frac{\partial T_i}{\partial t} = \Delta T_i + \frac{Q_i}{k_i}, \ r \in (r_{i-1}, r_i), \ 0 \leqslant \varphi \leqslant 2\pi, \ 0 < z < l, \ t > 0, \tag{1}$$

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}, \quad a_i^2 = \frac{c_i \rho_i}{k_i} = \text{const},$$

the initial condition and the boundary conditions

$$T(r, \varphi, z, 0) = f(r, \varphi, z),$$

$$\frac{\partial T_i}{\partial r} + \alpha_i [T_i - \psi_i(\varphi, z, t)] = 0 \ \text{at} \ r = r_j, \tag{2}$$

$$\alpha_1 = -h_1, \ \alpha_2 = h_2, \ j = \begin{cases} 0, & \text{if } i = 1, \\ 2, & \text{if } i = 2, \end{cases} \tag{3}$$

$$T(r, \varphi, z, t) = T(r, \varphi + 2\pi, z, t), \tag{4}$$

$$\frac{\partial T_i}{\partial z} = -\frac{1}{k_i} \chi_i(r, \varphi, t) \ \text{at} \ z = 0,$$

$$T_i = \chi_{i+2}(r, \varphi, t) \ \text{at} \ z = l, \tag{5}$$

also the contiguity conditions

$$T_1 = T_2, \ k_1 \frac{\partial T_1}{\partial r} - k_2 \frac{\partial T_2}{\partial r} = S \ \text{at} \ r = r_1. \tag{6}$$

Functions χ_i and χ_{i+2} must satisfy the contiguity conditions

$$\chi_{q+1} = \chi_{q+2}, \ k_1 \frac{\partial \chi_{q+1}}{\partial r} = k_2 \frac{\partial \chi_{q+2}}{\partial r} \quad \text{at} \quad r = r_1 \ (q = 0;\ 2). \tag{7}$$

As is well known, finite integral transformations are applied to independent variables within finite intervals. The variables r, φ, z in this problem can be eliminated by a finite integral transformation which will help to reduce the problem to a boundary-value problem for an ordinary differential equation.

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According to the general theory of finite integral transformations, the transform kernel will be eigenfunctions of the corresponding Sturm–Liouville problem:

$$R_{mnp}(r)\cos\sigma_m z\,(A_{mnp}\cos n\varphi + B_{mnp}\sin n\varphi), \tag{8}$$

where

$$R_{mnp}(r) = \frac{v_{i,n}(\omega_{i,mnp}r)}{v_{i,n}(\omega_{i,mnp}r_1)} \quad \text{at} \quad r\in[r_{i-1},\ r_i], \tag{9}$$

$$\begin{gathered} v_{i,n}(\omega_{i,mnp}r) = \left[N'_n(\omega_{i,mnp}r_j) + \beta_i N_n(\omega_{i,mnp}r_j)\right] J_n(\omega_{i,mnp}r) \\ - \left[J'_n(\omega_{i,mnp}r_j) + \beta_i J_n(\omega_{i,mnp}r_j)\right] N_n(\omega_{i,mnp}r), \\ \beta_i = \frac{\alpha_i}{\omega_{i,mnp}}, \end{gathered} \tag{10}$$

J_n and N_n are Bessel functions and Neumann functions of the n-th order,

$$\begin{gathered} \omega^2_{i,mnp} = \frac{c_i\rho_i}{k_i}\lambda^2_{mnp} - \sigma^2_m \quad \text{at} \quad r\in(r_{i-1},\ r_i), \\ \sigma_m = \frac{\pi(1+2m)}{2l}, \end{gathered} \tag{11}$$

λ_{mnp} are roots of the transcendental equation

$$k_1\omega_{1,mnp}\frac{v'_{1,n}(\omega_{1,mnp}r_1)}{v_{1,n}(\omega_{1,mnp}r_1)} = k_2\omega_{2,mnp}\frac{v'_{2,n}(\omega_{2,mnp}r_1)}{v_{2,n}(\omega_{2,mnp}r_1)}, \tag{12}$$

and A_{mnp}, B_{mnp} are integration constants. The prime sign indicates a derivative with respect to $(\omega_{i,mnp}r)$. Functions $R_{mnp}(r)$ with the weight factor r_μ are orthogonal on the interval $[r_0, r_2]$:

$$\int_{r_0}^{r_2} r\mu R_{mnp_1}R_{mnp_2}\,dr = 0, \quad \text{if} \quad p_1\neq p_2, \tag{13}$$

where $\mu = c_i\rho_i$, $r\in[r_{i-1}, r_i]$, $r\neq r_1$, $0\le\varphi\le 2\pi$, $0\le z\le l$.

Concerning any function $V_n(\lambda x)$ which satisfies the Bessel equation, the square of its norm on the interval $[a, b]$ is calculated as

$$\int_a^b xV_n^2(\lambda x)\,dx = \frac{x^2}{2}\left\{[V'_n(\lambda x)]^2 + \left[1-\frac{n^2}{(\lambda x)^2}\right]V_n^2(\lambda x)\right\}\Bigg|_a^b. \tag{14}$$

Using this formula and (12), we obtain

$$\begin{gathered} \|R_{mnp}\|^2 = \frac{r_1\left[v'_{1,n}(\omega_{1,mnp}r_1)\right]^2}{2v^2_{1,n}(\omega_{1,mnp}r_1)}\left(\varkappa_1 + \varkappa_2\frac{k_1^2\omega^2_{1,mnp}}{k_2^2\omega^2_{2,mnp}}\right) \\ + \sum_{i=1}^{2}\left[\frac{\varkappa_i r_1^2}{2}\left(1-\frac{n^2}{\omega^2_{i,mnp}r_1^2}\right) + \frac{2\varkappa_i}{\pi^2 r_j^2\omega^4_{i,mnp}v^2_{i,n}(\omega_{i,mnp}r_1)}\left(n^2 - h_i^2r_j^2 - \omega^2_{i,mnp}r_j^2\right)\right], \\ \varkappa_1 = c_1\rho_1,\ \varkappa_2 = -c_2\rho_2. \end{gathered} \tag{15}$$

The transformation formulas are found by expanding the functions into series in terms of the orthogonal functions in the respective Sturm–Liouville problem. With the aid of (13) and (15), considering the orthogonality of trigonometric functions, we obtain

$$T = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\frac{2}{\varepsilon_n\pi l\|R_{mnp}\|^2}R_{mnp}(r)\cos\sigma_m z[\overline{T}_{m,2n,p}\cos n\varphi + \overline{T}_{m,2n-1,p}\sin n\varphi], \tag{16}$$

where the summation applies to the roots of Eq. (12):

$$\begin{gathered} \overline{T}_{m\gamma p}(t) = \int_{r_0}^{r_2}\int_0^{2\pi}\int_0^l r\mu TW_{m\gamma p}\,dr\,d\varphi\,dz, \\ K_\gamma(\varphi) = \begin{cases}\cos n\varphi, & \text{if} \quad \gamma = 2n, \\ \sin n\varphi, & \text{if} \quad \gamma = 2n-1, \quad (n = 0,\ 1,\ 2,\ \ldots)\end{cases} \\ \varepsilon_n = \begin{cases} 2 & \text{at} \quad n = 0, \\ 1 & \text{at} \quad n\neq 0. \end{cases} \end{gathered} \tag{17}$$

We now define the finite integral transformation by expression (17) with the weight factor $r\mu$, where

$$W_{m\gamma p} = R_{mnp}(r)\cos\sigma_m z K_\gamma(\varphi) \tag{18}$$

is the transform kernel.

Equations (1) can be written as

$$\frac{\partial T}{\partial t} = \frac{1}{\mu}\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\cdot\frac{\partial}{\partial\varphi}\left(k\frac{\partial T}{\partial\varphi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right] + \frac{Q}{k\mu}, \tag{19}$$

$$r \in (r_0, r_2),\ r \neq r_1,\ 0 \leqslant \varphi \leqslant 2\pi,\ 0 < z < l,\ t > 0.$$

We multiply each term in the differential equation (19) by the transform kernel (18) with the weight factor $r\mu$, then integrate with respect to r from r_0 to r_2, with respect to φ from 0 to 2π, and with respect to z from 0 to l, whereupon we change the order of differentiation and integration on the left-hand side, which will finally yield

$$\frac{d\bar{T}_{m\gamma p}(t)}{dt} = \int_{r_0}^{r_2}\int_0^{2\pi}\int_0^l \frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) W_{m\gamma p}\,dr\,d\varphi\,dz$$

$$+ \int_{r_0}^{r_2}\int_0^{2\pi}\int_0^l \frac{1}{r}\cdot\frac{\partial}{\partial\varphi}\left(k\frac{\partial T}{\partial r}\right) W_{m\gamma p}\,dr\,d\varphi\,dz + \int_{r_0}^{r_2}\int_0^{2\pi}\int_0^l r\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) W_{m\gamma p}\,dr\,d\varphi\,dz + Q_{m\gamma p}(t), \tag{20}$$

where

$$Q_{m\gamma p}(t) = \int_{r_0}^{r_2}\int_0^{2\pi}\int_0^l \frac{r}{k} Q W_{m\gamma p}\,dr\,d\varphi\,dz. \tag{21}$$

In the first integral on the right-hand side of (20) we next divide the interval of integration with respect to r into segments $[r_0, r_1]$ and $[r_1, r_2]$, whereupon with the aid of (18) and (9) we obtain

$$\int_{r_0}^{r_2}\int_0^{2\pi}\int_0^l \frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) W_{m\gamma p}\,dr\,d\varphi\,dz = \int_0^l \cos\sigma_m z\left\{\int_0^{2\pi} K_\gamma(\varphi)\left[\sum_{i=1}^{2}\frac{k_i}{v_{i,n}(\omega_{i,mnp}r_1)}\int_{r_{i-1}}^{r_i}\frac{\partial}{\partial r}\right.\right.$$

$$\left.\left.\times\left(r\frac{\partial T_i}{\partial r}\right)v_{i,n}(\omega_{i,mnp}r)\,dr\right]d\varphi\right\}dz. \tag{22}$$

Twice integrating by parts over the intervals inside the brackets will yield

$$\sum_{i=1}^{2}\frac{k_i}{v_{i,n}(\omega_{i,mnp}r_1)}\int_{r_{i-1}}^{r_i}\frac{\partial}{\partial r}\left(r\frac{\partial T_i}{\partial r}\right)v_{i,n}(\omega_{i,mnp}r)\,dr = r_1\left[k_1\frac{\partial T_1(r_1,\varphi,z,t)}{\partial r} - k_2\frac{\partial T_2(r_1,\varphi,z,t)}{\partial r}\right]$$

$$-\frac{k_1 r_1 v_{1,n}(\omega_{1,mnp}r_0)}{v_{1,n}(\omega_{1,mnp}r_1)}\left[\frac{\partial T_1(r_0,\varphi,z,t)}{\partial r} + \alpha_1 T_1(r_0,\varphi,z,t)\right] + \frac{k_2 r_2 v_{2,n}(\omega_{2,mnp}r_2)}{v_{2,n}(\omega_{2,mnp}r_1)}\left[\frac{\partial T_2(r_2,\varphi,z,t)}{\partial r} + \alpha_2 T_2(r_2,\varphi,z,t)\right]$$

$$- r_1\left[k_1\omega_{1,mnp}\frac{v'_{1,n}(\omega_{1,mnp}r_1)}{v_{1,n}(\omega_{1,mnp}r_1)}T_1(r_1,\varphi,z,t) - k_2\omega_{2,mnp}\frac{v'_{2,n}(\omega_{2,mnp}r_1)}{v_{2,n}(\omega_{2,mnp}r_1)}T_2(r_1,\varphi,z,t)\right]$$

$$+\sum_{i=1}^{2}\frac{k_i}{v_{i,n}(\omega_{i,mnp}r_1)}\int_{r_{i-1}}^{r_i} T_i\frac{d}{dr}\left[r\frac{dv_{i,n}(\omega_{i,mnp}r)}{dr}\right]dr, \tag{23}$$

where

$$\alpha_i = -\frac{\omega_{i,mnp}v'_{i,n}(\omega_{i,mnp}r_j)}{v_{i,n}(\omega_{i,mnp}r_j)}.$$

Taking into consideration (3), (6), (12), and that functions $v_{i,n}$ satisfy the Bessel equations, we find that expression (23) is equal to

$$\psi_{mnp}(\varphi, z, t) = \sum_{i=1}^{2}\frac{k_i\omega^2_{i,mnp}}{v_{i,n}(\omega_{i,mnp}r_1)}\int_{r_{i-1}}^{r_i} rT_i v_{i,n}(\omega_{i,mnp}r)\,dr + n^2\int_{r_0}^{r_2}\frac{k}{r}TR_{mnp}\,dr, \tag{24}$$

$$\psi_{mnp}(\varphi, z, t) = r_1 S + \sum_{i=1}^{2} \frac{k_i h_i r_j v_{i,n}(\omega_{i,mnp} r_j)}{v_{i,n}(\omega_{i,mnp} r_1)} \psi_i. \tag{25}$$

Inserting expression (24) into (22) yields

$$\int_{r_0}^{r_2}\int_0^{2\pi}\int_0^{l} \frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) W_{m\gamma p}\, dr d\varphi dz = n^2 \int_{r_0}^{r_2}\int_0^{2\pi}\int_0^{l} \frac{k}{r} T W_{m\gamma p}\, dr d\varphi dz + \Psi_{m\gamma p}(t) - \sum_{i=1}^{2} k_i \omega_{i,mnp}^2 \int_{r_{i-1}}^{r_i}\int_0^{2\pi}\int_0^{l} r T_i W_{i,m\gamma p}\, dr d\varphi dz, \tag{26}$$

$$\Psi_{m\gamma p}(t) = \int_0^{2\pi}\int_0^{l} \psi_{mnp}(\varphi, z, t) \cos\sigma_m z K_\nu(\varphi)\, d\varphi dz. \tag{27}$$

We treat the second and the third integral on the right-hand side of Eq. (20) analogously, i.e., we divide the integration interval into $[r_0, r_1]$ and $[r_1, r_2]$, then twice integrate by parts the second integral with respect to z and the third integral with respect to φ, which with (5) yields

$$\int_{r_0}^{r_2}\int_0^{2\pi}\int_0^{l} \frac{1}{r}\cdot\frac{\partial}{\partial\varphi}\left(k\frac{\partial T}{\partial\varphi}\right) W_{m\gamma p}\, dr d\varphi dz = -n^2 \int_{r_0}^{r_2}\int_0^{2\pi}\int_0^{l} \frac{k}{r} T W_{m\gamma p}\, dr d\varphi dz, \tag{28}$$

$$\int_{r_0}^{r_2}\int_0^{2\pi}\int_0^{l} r\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) W_{m\gamma p}\, dr d\varphi dz = X_{m\gamma p}(t) - \sum_{i=1}^{2} k_i \sigma_m^2 \int_{r_{i-1}}^{r_i}\int_0^{2\pi}\int_0^{l} r T_i W_{i,m\gamma p}\, dr d\varphi dz, \tag{29}$$

where

$$X_{m\gamma p}(t) = \int_{r_0}^{r_2}\int_0^{2\pi} rk\chi(r, \varphi, t) R_{mnp}(r) K_\nu(\varphi)\, dr d\varphi, \tag{30}$$

$$\chi(r, \varphi, t) = \frac{1}{k_i}\chi_i(r, \varphi, t) + (-1)^m \sigma_m \chi_{i+2}(r, \varphi, t),\ r \in [r_{i-1}, r_i]. \tag{31}$$

Inserting the values of (26), (28), and (29) into (20), with (11) taken into consideration, we obtain an ordinary first-order differential equation in the transform variables:

$$\frac{d\overline{T}_{m\gamma p}(t)}{dt} + \lambda_{mnp}^2 \overline{T}_{m\gamma p}(t) = \Phi_{m\gamma p}(t), \tag{32}$$

where

$$\Phi_{m\gamma p}(t) = \Psi_{m\gamma p}(t) + X_{m\gamma p}(t) + Q_{m\gamma p}(t). \tag{33}$$

The solution to Eq. (32) is

$$\overline{T}_{m\gamma p}(t) = \exp(-\lambda_{mnp}^2)\left[D_{m\gamma p} + \int_0^t \exp(\lambda_{mnp}^2 \tau)\Phi_{m\gamma p}(\tau)\, d\tau\right]. \tag{34}$$

The integration constant $D_{m\gamma p}$ is determined from the initial condition (2), which becomes within the range of the transform variables

$$D_{m\gamma p} = \overline{T}_{m\gamma p}(0) = \int_{r_0}^{r_2}\int_0^{2\pi}\int_0^{l} r\mu f(r, \varphi, z) W_{m\gamma p}\, dr d\varphi dz. \tag{35}$$

Thus, the final solution to the problem is series (16) with $\overline{T}_{m\gamma p}(t)$ defined by expressions (34)-(35).

We will show here a few formulas of integral transformations and corresponding inverse transformations for the case of a two-layer hollow cylinder of finite length.

1. With boundary conditions of the first kind at the end surfaces

$$T_i|_{z=0} = \chi_i,\ T_i|_{z=l} = \chi_{i+2}, \tag{36}$$

the integral transformation is performed according to

$$\overline{T}_{m\gamma p}(t) = \int_{r_0}^{r_2}\int_0^{2\pi}\int_0^{l} r\mu T F_{m\gamma p}\, dr d\varphi dz, \tag{37}$$

$$F_{m\gamma p} = R_{mnp}(r) \sin\sigma_m z K_\nu(\varphi),\ \sigma_m = \frac{m\pi}{l}, \tag{38}$$

and the inverse transformation is

$$T=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\frac{2}{\varepsilon_n\pi l\,\|R_{mnp}\|^2}R_{mnp}(r)\sin\sigma_m z\,[\overline{T}_{m,2n,p}\cos n\varphi+\overline{T}_{m,2n-1,p}\sin n\varphi]. \tag{39}$$

2. With boundary conditions of the second kind at the end surfaces

$$\left.\frac{\partial T_i}{\partial z}\right|_{z=0}=-\frac{1}{k_i}\chi_i,\quad \left.\frac{\partial T_i}{\partial z}\right|_{z=l}=\frac{1}{k_i}\chi_{i+2}, \tag{40}$$

the integral transformation is performed according to

$$\overline{T}_{m\gamma p}(t)=\int_{r_0}^{r_2}\int_0^{2\pi}\int_0^{l} r\mu T W_{m\gamma p}\,dr\,d\varphi\,dz,\quad \sigma_m=\frac{m\pi}{l}, \tag{41}$$

and the inverse transformation is

$$T=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\frac{2}{\varepsilon_n\pi l\,\|R_{mnp}\|^2}R_{mnp}(r)\cos\sigma_m z\,[\overline{T}_{m,2n,p}\cos n\varphi+\overline{T}_{m,2n-1,p}\sin n\varphi]. \tag{42}$$

3. With boundary conditions of the first and the second kind at the end surfaces

$$\left.T_i\right|_{z=0}=\chi_i,\quad \left.\frac{\partial T_i}{\partial z}\right|_{z=l}=\frac{1}{k_i}\chi_{i+2}, \tag{43}$$

the integral transformation is performed according to

$$\overline{T}_{m\gamma p}(t)=\int_{r_0}^{r_2}\int_0^{2\pi}\int_0^{l} r\mu T F_{m\gamma p}\,dr\,d\varphi\,dz,\quad \sigma_m=\frac{\pi(1+2m)}{2l}, \tag{44}$$

and the inverse transformation is

$$T=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\frac{2}{\varepsilon_n\pi l\,\|R_{mnp}\|^2}R_{mnp}(r)\sin\sigma_m z\,[\overline{T}_{m,2n,p}\cos n\varphi+\overline{T}_{m,2n-1,p}\sin n\varphi]. \tag{45}$$

The summations in (39), (42), and (45) apply to the roots of Eq. (12). Instead of condition (5) in problem (1)-(7), in these three cases we consider condition (36), (40), and (43) respectively. Similar formulas for a homogeneous hollow cylinder of infinite length were given in [1].

NOTATION

T_i is the temperature of the i-th cylinder, (i = 1, 2);
r, φ, z are the cylindrical coordinates;
t is the time;
a_i^{-1} are the thermal diffusivity coefficients;
k_i are the thermal conductivity coefficients;
h_i are the coefficients of external heat transfer;
l is the length of cylinder;
r_0 is the inside radius of inner cylinder;
r_1 is the radius of interface between cylinders;
r_2 is the outside radius of outer cylinder.

LITERATURE CITED

1. Sh. N. Plyat, Inzh.-Fiz. Zh., No. 6 (1962).
2. B. M. Budak, A. A. Samarskii, and A. N. Tikhonov, Collection of Problems in Mathematical Physics [in Russian], Moscow (1956).
3. G. A. Grinberg, Selected Problems in the Mathematical Theory of Electrical and Magnetic Phenomena [in Russian], Izd. Akad. Nauk SSSR, Moscow–Leningrad (1948).
4. N. S. Koshlyakov, É. B. Gliner, and M. M. Smirnov, Partial Differential Equations in Mathematical Physics [in Russian], Izd. Vysshaya Shkola, Moscow (1970).
5. V. N. Gavdzinskii, Research in Heat Conduction (collection of articles) [in Russian], Minsk (1967).

6. K. J. Tranter, Integral Transformations in Mathematical Physics [Russian translation], Izd. GITTL (1956).
7. Sneddon, Fourier Transformations [Russian translation], Izd. Inostr. Lit. (1955).